Equilibrium crystal shape of the Potts model at the first-order transition point

1997 J. Phys. A: Math. Gen. 30 3779

(http://iopscience.iop.org/0305-4470/30/11/011)

J. Phys. A: Math. Gen. **30** (1997) 3779–3793. Printed in the UK PII: S0305-4470(97)76796-9

Equilibrium crystal shape of the Potts model at the first-order transition point

Masafumi Fujimoto

Department of Physics, Nara Medical University, Kashihara, Nara 634, Japan

Received 30 July 1996, in final form 18 March 1997

Abstract. We investigate the Q-state two-dimensional Potts model. The anisotropic interfacial tension is related by duality to the anisotropic correlation length. For $Q > 4$ we calculate exactly the anisotropic correlation length at the first-order transition point. From the calculated anisotropic correlation length, the equilibrium crystal shape (ECS) is derived via the Wulff construction. The ECS is expressed by means of a simple algebraic curve. Regarding Q as a temperature scale, we show that the Potts model has the same ECS as the eight-vertex model. We discuss a connection between the ECS and the q-deformed pseudo-Euclidian algebra.

1. Introduction

Thermal evolution of the equilibrium crystal shape (ECS) has attracted much attention in relation to the roughening transition phenomena [1–16]. Theoretically, if the interfacial tension is known with its full directional dependence (or anisotropy), we can determine the ECS via the Wulff construction [1–4]. The problem reduces to the calculation of the anisotropic interfacial tension. Finding reliable values of the interfacial tension for all directions is still very difficult in most cases. The first exact analysis of the ECS was done for the square-lattice (nearest-neighbour) Ising model [7–9] (see also [10–13]). For the square-lattice Ising model some authors [17–19] have proven that the anisotropic interfacial tension is related by duality [20–23] to the anisotropic correlation length. The anisotropic correlation length has been calculated by the Pfaffian method [24, 25]. From these results the ECS was derived with the help of the Wulff construction.

In a previous paper [26] we calculated the anisotropic interfacial tension of the eight-vertex model by a new method [27, 28] which introduces the shift operator into the standard transfer matrix argument. From the calculated anisotropic interfacial tension, we obtained the ECS via the Wulff construction. The ECS of the eight-vertex model is represented as a simple algebraic curve

$$\alpha^2\beta^2 + 1 + A_3(\alpha^2 + \beta^2) + A_4\alpha\beta = 0 \tag{1.1a}$$

where

$$\alpha = \exp(-\Lambda X/k_{\mathrm{B}}T), \qquad \beta = \exp(-\Lambda Y/k_{\mathrm{B}}T) \tag{1.1b}$$

with a scale factor Λ and the position vector (X, Y) of a point on the ECS. The eight-vertex model reduces to the square-lattice Ising model and the six-vertex model in special limits with respect to a parameter q [29]. Showing that the coefficients A_3 and A_4 are independent of q, we extended the fact that the ECSs of the Ising model and the six-vertex model are essentially the same [5–9, 13].

0305-4470/97/113779+15$19.50

Equation (1.1a) is a symmetric biquadratic relation between α and β. The symmetric biquadratic relation is naturally parametrized in terms of the Jacobian elliptic functions (see [29, chapter 15]). In the analysis of the eight-vertex model [26] we showed that an elliptic function $p(v)$ in the expression of the interfacial tension corresponds to the sn function in this parametrization. Thus, $p(v)$ in the expression of the interfacial tension reflects the ECS (1.1). Besides the eight-vertex model, the interfacial tension of several solvable models was calculated along a special direction [30–32]. We note that elliptic functions in the expressions of the interfacial tension of these solvable models are (essentially) the sn function. It is suggested that their ECSs are also the symmetric biquadratic relation (1.1). For hard squares with diagonal attractions this fact was proven by direct calculations [33]. We expect that (1.1) is a universal shape which appears as the ECSs of a wide class of models. Further study of (1.1) is desirable.

Here, we consider the Q-state Potts model [29, 34–35]. Consider a square lattice. A spin variable σ_{jk}, which takes values in the group $\mathbb{Z}_Q = \{0, 1, \ldots, Q-1\}$ with the addition mod Q as a group law, is associated with each site (j, k). The Hamiltonian of the Potts model is defined by

$$E = -J_1 \sum_{j,k} \delta(\sigma_{j,k+1} - \sigma_{jk}) - J_2 \sum_{j,k} \delta(\sigma_{j+1,k} - \sigma_{jk}) \tag{1.2}$$

where nearest-neighbour spins are coupled by J_1 if they are horizontal neighbours, by J_2 if they are vertical neighbours, and

$$\delta(\sigma - \sigma') = \begin{cases} 1 & \text{if } \sigma - \sigma' = 0 \\ 0 & \text{if } \sigma - \sigma' \neq 0. \end{cases} \tag{1.3}$$

We assume that $J_1, J_2 > 0$. The partition function is given by

$$Z = \sum_{\sigma} \exp\left[K_1 \sum_{j,k} \delta(\sigma_{j,k+1} - \sigma_{jk}) + K_2 \sum_{j,k} \delta(\sigma_{j+1,k} - \sigma_{jk}) \right] \tag{1.4}$$

where the outer sum is over all the spin configurations and

$$K_1 = J_1/k_{\mathrm{B}}T \qquad K_2 = J_2/k_{\mathrm{B}}T. \tag{1.5}$$

For later convenience we rewrite (1.4) as

$$Z = \sum_{\sigma} \prod_{j,k} V_1(\sigma_{j,k+1} - \sigma_{jk}) V_2(\sigma_{j+1,k} - \sigma_{jk}) \tag{1.6}$$

with

$$\begin{aligned} V_1(x) &= (\mathrm{e}^{K_1} - 1)\delta(x) + 1 \\ V_2(y) &= (\mathrm{e}^{K_2} - 1)\delta(y) + 1. \end{aligned} \tag{1.7}$$

The $Q = 2$ Potts model is equivalent to the Ising model. For general Q it is known that the Potts model is solvable at the phase transition point [29, 36]. When $Q > 4$, the phase transition is first order. Along the horizontal direction, Laanait [39] showed that the interfacial tension is related by duality to the correlation length. At the first-order transition point the correlation length was calculated along special directions [37–38]. It is noted that the correlation length is expressed in terms of the sn function there. In this paper, we calculate the exact ECS of the Potts model for $Q > 4$ at the first-order transition point. We examine connections between the ECS of the Potts model and the algebraic curve (1.1).

The format of the present paper is as follows. In section 2, using duality transformations, we derive a relation between the anisotropic interfacial tension and the anisotropic correlation length. In section 3 we consider the case $Q > 4$. The anisotropic correlation

length is calculated exactly at the first-order transition point. From the calculated anisotropic correlation length, the ECS is found via the Wulff construction. In section 4 we discuss the energy–momentum excitations of a spin chain which is associated with the Potts model. Section 5 is devoted to a summary and discussion.

2. Duality of interfacial tension and correlation length

For the Q-state Potts model we derive a relation between the anisotropic interfacial tension and the anisotropic correlation length. The phase transition point of the Potts model occurs when

$$x_1 x_2 = 1 \tag{2.1}$$

with

$$x_j = Q^{-1/2}(\mathrm{e}^{K_j} - 1) \qquad j = 1, 2 \tag{2.2}$$

(see [29, chapter 12]). We start by assuming that the system is in the ferromagnetic ordered state: $x_1 x_2 > 1$. Consider a square lattice Λ_1 of $2N + 2$ columns and $2M + 2$ rows ($M \gg N \gg 1$) (figure 1). We impose on it two types of boundary conditions:

(a) The upper boundary spins are fixed to be Q_0, and the lower boundary spins $Q_1 (\neq Q_0)$; on the left boundary $\sigma_{j,-N} = Q_1$ if $j \leqslant -L$, and $\sigma_{j,-N} = Q_0$ otherwise; on the right boundary $\sigma_{j,N+1} = Q_1$ if $j \leqslant L$, and $\sigma_{j,N+1} = Q_0$ otherwise.

(b) All the boundary spins are fixed to be Q_0.

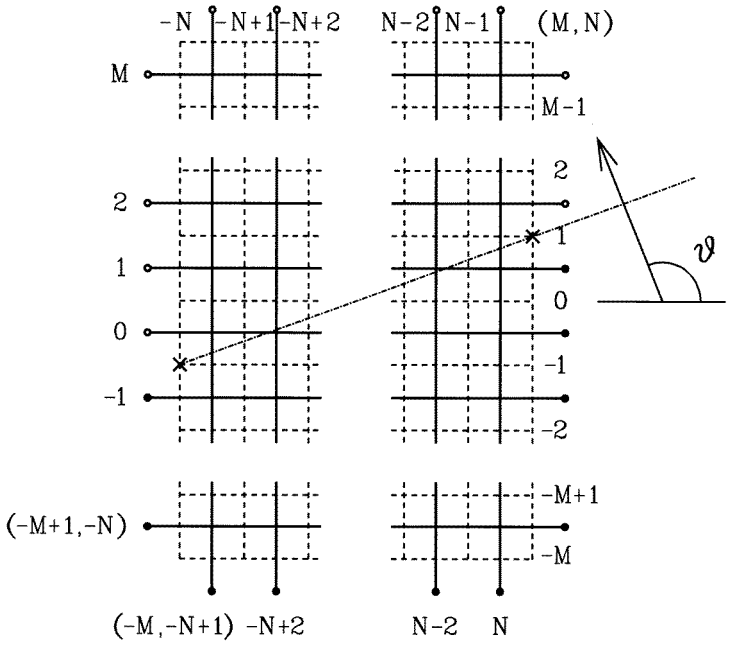

Figure 1. The square lattice, Λ_1, is shown by full lines. The boundary conditions (a) are imposed on it: the boundary spins denoted by open circles (respectively full circles) are fixed to be Q_0 (respectively Q_1). Broken lines represent the dual lattice Λ_1^*.

We denote by γ the interfacial tension between two ordered phases. The interfacial tension γ along the direction designated by $L/N = -1/\tan\theta$ $(0 < \theta < \pi)$ is defined by

$$-\gamma(\theta)/k_{\mathrm{B}}T = \lim_{L,M,N\to\infty} (2N)^{-1} \sin\theta \ln Z^{\mathrm{a}}_{\Lambda_1}/Z^{\mathrm{b}}_{\Lambda_1} \tag{2.3}$$

where $Z^{\mathrm{a}}_{\Lambda_1}$ (respectively $Z^{\mathrm{b}}_{\Lambda_1}$) is the partition function of the system with the boundary conditions (a) (respectively (b)); the limit is taken with the ratio L/N fixed and under the condition $M \gg N$.

In the case of isotropic interactions $K_1 = K_2$ and along the horizontal direction $\theta = \pi/2$ Laanait [39] showed that γ is related to the correlation length ξ^* at the dual temperature. The argument is extended into general cases of anisotropic interactions and general directions. We define the dual model as follows. A spin variable $\hat{\sigma}_{jk} \in \mathbb{Z}_Q$ is placed on each site (j,k) of the dual lattice Λ_1^*; the site (j,k) of Λ_1^* is connected with the site (j,k) of Λ_1 by shifting in both directions by a half-lattice spacing (figure 1). The interaction constants K_1^* and K_2^* on Λ_1^* are given by

$$\mathrm{e}^{K_1^*} - 1 = Q/(\mathrm{e}^{K_2} - 1) \qquad \mathrm{e}^{K_2^*} - 1 = Q/(\mathrm{e}^{K_1} - 1). \tag{2.4}$$

(Note that (2.1) corresponds to the self-dual condition.)

To relate calculations of the dual model to those of the original model, we introduce the transformation

$$\begin{aligned} \hat{V}_1(\hat{x}) &= \sum_{y=0}^{Q-1} V_2(y)\mathrm{e}^{-2\pi \mathrm{i} y\hat{x}/Q} = (\mathrm{e}^{K_2} - 1)[(\mathrm{e}^{K_1^*} - 1)\delta(\hat{x}) + 1] \\ \hat{V}_2(\hat{y}) &= \sum_{x=0}^{Q-1} V_1(x)\mathrm{e}^{2\pi \mathrm{i} x\hat{y}/Q} = (\mathrm{e}^{K_1} - 1)[(\mathrm{e}^{K_2^*} - 1)\delta(\hat{y}) + 1] \end{aligned} \tag{2.5}$$

(see, for example, [23, 35].) The inverse transformation gives

$$\begin{aligned} V_1(\sigma_{j,k+1} - \sigma_{jk}) &= Q^{-1} \sum_{\Delta^{(2)}_{jk}=0}^{Q-1} \hat{V}_2(\Delta^{(2)}_{jk}) \exp\left[\frac{2\pi \mathrm{i}}{Q}(\sigma_{j,k+1} - \sigma_{jk})\Delta^{(2)}_{jk}\right] \\ V_2(\sigma_{j+1,k} - \sigma_{jk}) &= Q^{-1} \sum_{\Delta^{(1)}_{jk}=0}^{Q-1} \hat{V}_1(\Delta^{(1)}_{jk}) \exp\left[-\frac{2\pi \mathrm{i}}{Q}(\sigma_{j+1,k} - \sigma_{jk})\Delta^{(1)}_{jk}\right]. \end{aligned} \tag{2.6}$$

Using (2.6) in (1.6), we obtain

$$Z^{\mathrm{b}}_{\Lambda_1} = Q^{-(4MN+2M+2N)} {\sum_{\Delta}}' \prod_{j,k} \hat{V}_1(\Delta^{(1)}_{jk})\hat{V}_2(\Delta^{(2)}_{jk}) \tag{2.7}$$

where the summation is over configurations of Δ which satisfy the condition

$$\Delta^{(1)}_{jk} - \Delta^{(2)}_{jk} - \Delta^{(1)}_{j-1,k} + \Delta^{(2)}_{j,k-1} \equiv 0 \qquad (\mathrm{mod}\, Q) \tag{2.8}$$

for $-M + 1 \leqslant j \leqslant M$ and $-N + 1 \leqslant k \leqslant N$.

The configurations of Δ are connected with those of $\hat{\sigma}$ by

$$\begin{aligned} \Delta^{(1)}_{jk} &\equiv \hat{\sigma}_{jk} - \hat{\sigma}_{j,k-1} \qquad (\mathrm{mod}\, Q) \\ \Delta^{(2)}_{jk} &\equiv \hat{\sigma}_{jk} - \hat{\sigma}_{j-1,k} \qquad (\mathrm{mod}\, Q). \end{aligned} \tag{2.9}$$

Then, (2.8) is satisfied automatically. It follows that

$$Z^{\mathrm{b}}_{\Lambda_1} = Q^{-(2M+1)(2N+1)}(\mathrm{e}^{K_1} - 1)^{2M(2N+1)}(\mathrm{e}^{K_2} - 1)^{(2M+1)2N} Z^{\mathrm{f}}_{\Lambda_1^*} \tag{2.10}$$

where $Z^{\mathrm{f}}_{\Lambda_1^*}$ is the partition function of the dual model with free boundary conditions. Similarly, we find that

$$Z^{\mathrm{a}}_{\Lambda_1} = Q^{-(2M+1)(2N+1)}(\mathrm{e}^{K_1}-1)^{2M(2N+1)}(\mathrm{e}^{K_2}-1)^{(2M+1)2N} Z^{\mathrm{f}}_{\Lambda_1^*} \times \langle [Q\delta(\hat{\sigma}_{-L,-N} - \hat{\sigma}_{LN}) - 1]/(Q-1)\rangle^{\mathrm{f}}_{\Lambda_1^*} \tag{2.11}$$

where $\langle\cdot\rangle$ denotes the expectation value. From (2.10) and (2.11), it follows that

$$Z^{\mathrm{a}}_{\Lambda_1}/Z^{\mathrm{b}}_{\Lambda_1} = \langle [Q\delta(\hat{\sigma}_{-L,-N} - \hat{\sigma}_{LN}) - 1]/(Q-1)\rangle^{\mathrm{f}}_{\Lambda_1^*}. \tag{2.12}$$

In the infinite volume limit $\Lambda_1^* \to \mathbb{Z}^2$ we define the correlation length ξ^* along the direction θ by

$$-1/\xi^* = \lim_{r\to\infty} r^{-1} \ln\langle [Q\delta(\hat{\sigma}_{-l,-n} - \hat{\sigma}_{ln}) - 1]/(Q-1)\rangle_{\mathbb{Z}^2} \tag{2.13}$$

where $r = 2(l^2+n^2)^{1/2}$ and the limit $r \to \infty$ is taken with the ratio $l/n(= -1/\tan\theta)$ fixed. Set $L = (2k+1)l$ and $N = (2k+1)n$ in (2.12). Lemma 2.1 (a) of [39] shows that

$$\frac{1}{(2k+1)r} \ln Z^{\mathrm{a}}_{\Lambda_1}/Z^{\mathrm{b}}_{\Lambda_1} = \frac{1}{(2k+1)r} \ln\langle [Q\delta(\hat{\sigma}_{-L,-N} - \hat{\sigma}_{LN}) - 1]/(Q-1)\rangle^{\mathrm{f}}_{\Lambda_1^*} \geqslant \frac{1}{(2k+1)r} \sum_{j=-k}^{k} \ln\langle [Q\delta(\hat{\sigma}_{(2j-1)l,(2j-1)n} - \hat{\sigma}_{(2j+1)l,(2j+1)n}) - 1]/(Q-1)\rangle^{\mathrm{f}}_{\Lambda_1^*}. \tag{2.14}$$

Take the $M, k \to \infty$ limit with l and n fixed to be constants. On the r.h.s. of (2.14) we repeat the same argument used in the proof of Theorem 2 (i) of [40]. It is found that

$$-\gamma/k_{\mathrm{B}}T \geqslant r^{-1} \ln\langle [Q\delta(\hat{\sigma}_{-l,-n} - \hat{\sigma}_{ln}) - 1]/(Q-1)\rangle_{\mathbb{Z}^2} \tag{2.15}$$

where γ is the interfacial tension along the direction designated by the ratio l/n. We take the $r \to \infty$ limit with the ratio l/n fixed on the r.h.s. of (2.15). It follows that

$$\gamma/k_{\mathrm{B}}T \leqslant 1/\xi^*. \tag{2.16}$$

For fixed L and N the expectation value on the r.h.s. of (2.12) is monotone increasing in Λ_1^*, which is a consequence of Ginibre inequalities [41] (see also [42]). The monotonicity shows that

$$R^{-1} \ln Z^{\mathrm{a}}_{\Lambda_1}/Z^{\mathrm{b}}_{\Lambda_1} \leqslant R^{-1} \ln\langle [Q\delta(\hat{\sigma}_{-L,-N} - \hat{\sigma}_{LN}) - 1]/(Q-1)\rangle_{\mathbb{Z}^2} \tag{2.17}$$

with $R = 2(L^2+N^2)^{1/2}$. Taking $L, M, N \to \infty$ limit with the ratio L/N fixed gives the converse inequality of (2.16). Thus, we obtain

$$\gamma/k_{\mathrm{B}}T = 1/\xi^* \tag{2.18}$$

for all θ [7–9, 19].

3. Equilibrium crystal shape

In this section we consider the case $Q > 4$, where the Potts model has a first-order phase transition [29, 36]. The exact ECS is calculated at the first-order transition point. Klümper *et al* [37] (KSZ) obtained the correlation length along the diagonal direction there. When the interactions are isotropic, Buffenoir and Wallon [38] calculated it along the vertical direction. In subsection 3.1, using the shift operator, we extend the analysis by KSZ to find the anisotropic correlation length. In subsection 3.2 the ECS is derived from the anisotropic correlation length with the help of the relation (2.18) and the Wulff construction.

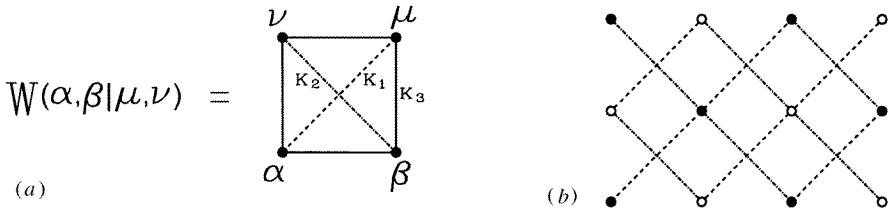

Figure 2. (*a*) The Boltzmann weights, W, around a face. The two-spin interactions, K_1 and K_2, are represented by a broken line and a chain line, respectively, and the four-spin interaction, K_3, by full lines. (*b*) In the decoupling limit $ab = cd$ (or $K_3 = 0$) the IRF model factors into two independent Potts models, which are shown by open circles and full circles.

3.1. Anisotropic correlation length

KSZ investigated a square-lattice model with interactions round faces (IRF) [29]. The IRF model is defined as follows. We locate a spin variable $\alpha_j \in \mathbb{Z}_Q$ at each site j of a square lattice. The Boltzmann weight is assigned on each face depending on spin configurations around it. When spins around a face are α, β, μ, and ν counterclockwise starting from the southwest corner (figure 2(*a*)), the Boltzmann weight around it is

$$W(\alpha, \beta|\mu, \nu) = \exp[-\varepsilon(\alpha, \beta|\mu, \nu)/k_{\mathrm{B}}T] \tag{3.1}$$

with

$$\begin{aligned}-\varepsilon(\alpha, \beta|\mu, \nu)/k_{\mathrm{B}}T &= K_0 + K_1\delta(\alpha - \mu) + K_2\delta(\beta - \nu) \\ &\quad + K_3\{\delta(\alpha - \mu)\delta(\beta - \nu) + [1 - \delta(\alpha - \mu)][1 - \delta(\beta - \nu)]\}.\end{aligned} \tag{3.2}$$

There are four different Boltzmann weights:

$$\begin{aligned} a &= W(\alpha, \beta|\alpha, \beta) = \mathrm{e}^{K_0+K_1+K_2+K_3} \\ b &= W(\alpha, \beta|\bar{\alpha}, \bar{\beta}) = \mathrm{e}^{K_0+K_3} \\ c &= W(\alpha, \beta|\alpha, \bar{\beta}) = \mathrm{e}^{K_0+K_1} \\ d &= W(\alpha, \beta|\bar{\alpha}, \beta) = \mathrm{e}^{K_0+K_2} \end{aligned} \tag{3.3}$$

where $\bar{\alpha}$ is one of the $Q - 1$ values which differ from α. When $Q = 2$, the IRF model is equivalent to the eight-vertex model. Using inversion relations, and borrowing arguments from the analyses of the eight-vertex model, KSZ determined the phase transition point for general Q. In the $ab = cd$ (or $K_3 = 0$) limit the IRF model factors into two independent Potts models (figure 2(*b*)). For $Q > 4$ KSZ calculated the correlation length of the Potts model along the diagonal direction at the self-dual (or first-order transition) point. We extend the calculation by KSZ to find the anisotropic correlation length of the Potts model.

Hereafter, analyses are restricted to the decoupling limit $ab = cd$. Moreover, we assume that $Q > 4$, and that the system is at the first-order transition point. At the first-order transition point the four Boltzmann weights in (3.3) are parametrized as

$$\begin{aligned} \mathrm{e}^{K_0} &= b = \sinh(\lambda/2 + u)\sinh(\lambda/2 - u) \\ \mathrm{e}^{K_1} &= a/d = \sinh(3\lambda/2 - u)/\sinh(\lambda/2 + u) \\ \mathrm{e}^{K_2} &= a/c = \sinh(3\lambda/2 + u)/\sinh(\lambda/2 - u) \end{aligned} \tag{3.4}$$

with

$$2\cosh\lambda = \sqrt{Q} \qquad \lambda > 0 \tag{3.5}$$

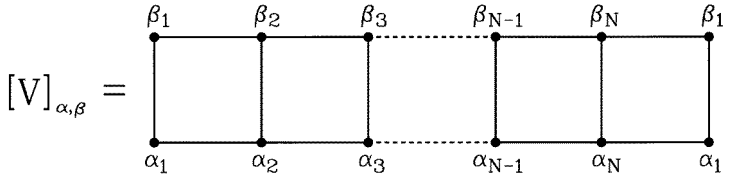

Figure 3. The one-parameter family of commuting transfer matrices.

and

$$-\lambda/2 < u < \lambda/2. \tag{3.6}$$

We suppose a square lattice Λ_2 of N columns and M rows with periodic boundary conditions in both directions (M, N even). Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ and $\beta = \{\beta_1, \beta_2, \ldots, \beta_N\}$ be the spin configurations on two successive rows. A one-parameter family of row-to-row transfer matrices is defined by

$$[\mathbf{V}(u)]_{\alpha,\beta} = [\sinh(\lambda)\sinh(2\lambda)]^{-N} \prod_{j=1}^{N} W(\alpha_j, \alpha_{j+1}|\beta_{j+1}, \beta_j|u) \tag{3.7}$$

where $\alpha_{N+1} = \alpha_1$ and $\beta_{N+1} = \beta_1$ (figure 3). The Boltzmann weights W satisfy the standard initial condition [29, 43–44]

$$W(\alpha, \beta|\mu, \nu| - \lambda/2) = \sinh(\lambda)\sinh(2\lambda)\delta(\alpha - \mu). \tag{3.8}$$

Because of the periodic boundary conditions, (3.8) implies that $\mathbf{V}(u)$ reduces to the shift operator in the $u = -\lambda/2$ limit. The Boltzmann weights W also satisfy the Yang–Baxter relation [29, 43–44]

$$\sum_{\nu=0}^{Q-1} W(\alpha, \beta'|\nu, \gamma'|u)W(\gamma', \nu|\alpha', \beta|u')W(\nu, \beta'|\gamma, \beta'|u'')$$
$$= \sum_{\nu=0}^{Q-1} W(\gamma', \alpha|\nu, \beta|u'')W(\alpha, \beta'|\gamma, \nu|u')W(\nu, \gamma|\alpha', \beta|u) \tag{3.9}$$

for all $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \in \mathbb{Z}_Q$ with $u' = u + u'' + \lambda/2$. The Yang–Baxter relation (3.9) shows that, for all complex numbers u and u', $\mathbf{V}(u)$ and $\mathbf{V}(u')$ commute with each other; therefore $\mathbf{V}(u)$ and $\mathbf{V}(u')$ can be diagonalized simultaneously. We denote the eigenvalues of $\mathbf{V}(u)$ by $V(u)$.

We can prove that the eigenvalue $V(u)$ satisfies the inversion relation

$$V(u)V(u+\lambda) = \left[\frac{\sinh(\lambda+u)\sinh(\lambda-u)}{\sinh^2(\lambda)}\right]^{2N} + \mathrm{O}(\mathrm{e}^{-\epsilon N}) \tag{3.10}$$

for large N with $\epsilon > 0$ (see also [45, 46].) Using the inversion relation (3.10), and assuming some analytic properties of the eigenvalues $V(u)$, we can determine their asymptotic forms as $N \to \infty$. For convenience we introduce a limiting function by

$$L(u) = \lim_{N\to\infty} V(u)/V_0(u) \tag{3.11}$$

where $V_0(u)$ is the largest eigenvalue in the regime (3.5), (3.6). KSZ obtained for the next-largest eigenvalues

$$L_1(u) = k\,\mathrm{snh}[2K(u-\Theta_1)/\pi]\,\mathrm{snh}[2K(u-\Theta_2)/\pi] \tag{3.12}$$

with imaginary free parameters Θ_1, Θ_2; the snh function is given by

$$\mathrm{snh}(v) = -\mathrm{i}\,\mathrm{sn}(\mathrm{i}v) \tag{3.13}$$

where sn is the Jacobian sn function to the modulus k which is defined by requiring that the corresponding quarter periods K, K' satisfy

$$K'/K = 2\lambda/\pi \tag{3.14}$$

(see appendix A of KSZ).

We denote by $G(m, n)$ the correlation between the sites $(0, 0)$ and (m, n) in the $\Lambda_2 \to \mathbb{Z}^2$ limit:

$$G(m, n) = \langle [Q\delta(\alpha_{00} - \alpha_{mn}) - 1]/(Q - 1)\rangle^{\mathrm{p}}_{\mathbb{Z}^2}. \tag{3.15}$$

The correlation length ξ of the Potts model is defined by

$$-1/\xi = \lim_{r\to\infty} r^{-1} \ln[G(m, n) - G_\infty] \qquad G_\infty = \lim_{r\to\infty} G(m, n) \tag{3.16}$$

where $r = (m^2 + n^2)^{1/2}/\sqrt{2}$ and the limit is taken with the ratio $n/m = -\tan(\theta + \pi/4)$ $(\pi/4 < \theta < 5\pi/4)$ fixed and under the condition that $m + n$ is even; note that, in the decoupling limit, $G(m, n) = 0$ if $m + n$ is odd (figure 2(*b*)).

KSZ investigated ξ along the diagonal direction $\theta = 3\pi/4$ as follows. In the standard transfer matrix method the correlation between the sites $(0, 0)$ and (m, n) is represented as

$$\langle [Q\delta(\alpha_{00} - \alpha_{mn}) - 1]/(Q - 1)\rangle^{\mathrm{p}}_{\Lambda_2} = \frac{\mathrm{Tr}[\mathbf{A}_0\mathbf{V}^m(u)\mathbf{A}^*_n\mathbf{V}^{M-m}(u)]}{\mathrm{Tr}[\mathbf{V}^M(u)]} \tag{3.17}$$

where $\mathbf{A}_j$ $(j = 0, 1, \ldots, M - 1)$ is a diagonal matrix given by

$$[\mathbf{A}_j]_{\alpha,\beta} = \begin{cases} \exp(-2\pi\alpha_j/Q) & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases} \tag{3.18}$$

and $\mathbf{A}^*_j$ is the complex conjugate of $\mathbf{A}_j$. Apply a similarity transformation which diagonalizes $\mathbf{V}(u)$. We take the $M \to \infty$ limit firstly, and then $N \to \infty$ limit. In the $M \to \infty$ limit, we find that

$$\langle [Q\delta(\alpha_{00} - \alpha_{mn}) - 1]/(Q - 1)\rangle^{\mathrm{p}}_{\Lambda_2} = \sum_p [\tilde{\mathbf{A}}_0]_{0,p}[\tilde{\mathbf{A}}^*_n]_{p,0}\left[\frac{V_p(u)}{V_0(u)}\right]^m \tag{3.19}$$

where $V_p(u)$ is the p-th eigenvalue of $\mathbf{V}(u)$ in decreasing order of magnitude and $\tilde{\mathbf{A}}_j$ is the matrix transformed from $\mathbf{A}_j$. Equation (3.19) shows that, when $n = 0$ and m becomes large, the correlation length ξ along the diagonal direction can be calculated from the ratios between the largest eigenvalue $V_0(u)$ and the next-largest eigenvalues. In the $N \to \infty$ limit these ratios are given by the limiting function in (3.12). Because of a continuous distribution of the next-largest eigenvalues, the summation on the r.h.s. of (3.19) is converted into an integral over the imaginary parameters Θ_1 and Θ_2. Estimating the integral by the method of steepest descent, KSZ obtained ξ along the diagonal direction.

To find the anisotropy of ξ, we consider the $m \to \infty$ limit with the ratio n/m fixed. In this limit contribution from the matrix elements $[\tilde{\mathbf{A}}^*_n]_{p,0}$ is important as well as the ratios between the eigenvalues on the r.h.s. of (3.19). Calculating the matrix elements is a very complicated problem, however. To overcome the difficulty, we introduce the shift operator $\mathbf{V}(-\lambda/2)$ [27, 28]. The shift operator connects $\mathbf{A}^*_n$ with $\mathbf{A}^*_0$ by

$$\mathbf{A}^*_n = \mathbf{V}(-\lambda/2)^{-n}\mathbf{A}^*_0\mathbf{V}(-\lambda/2)^n. \tag{3.20}$$

Substituting (3.20) into (3.17) gives

$$\langle[Q\delta(\alpha_{00}-\alpha_{mn})-1]/(Q-1)\rangle^{\mathrm{p}}_{\Lambda_2}=\sum_p[\tilde{\mathbf{A}}_0]_{0,p}[\tilde{\mathbf{A}}_0^*]_{p,0}\left[\frac{V_p(u)}{V_0(u)}\left(\frac{V_p(-\lambda/2)}{V_0(-\lambda/2)}\right)^{\eta}\right]^m \tag{3.21}$$

where $\eta=-n/m=\tan(\theta+\pi/4)$. Equation (3.21) implies that ξ along the direction designated by η can be found from the eigenvalues of the transfer matrix and those of the shift operator without calculating the matrix elements $[\tilde{\mathbf{A}}_n^*]_{p,0}$.

Using (3.12) in (3.21), we get

$$\begin{aligned}G(m,n)-G_\infty\sim&\int_0^{2\pi\mathrm{i}}\mathrm{d}\Theta_1\int_0^{2\pi\mathrm{i}}\mathrm{d}\Theta_2\,\rho(\Theta_1,\Theta_2)\\&\times\{k\,\mathrm{snh}[2K(u-\Theta_1)/\pi]\mathrm{snh}[2K(u-\Theta_2)/\pi]\\&\times\{k\,\mathrm{snh}[2K(-\lambda/2-\Theta_1)/\pi]\mathrm{snh}[2K(-\lambda/2-\Theta_2)/\pi]\}^{\eta}\}^m\end{aligned} \tag{3.22}$$

where $\rho(\Theta_1,\Theta_2)$ is to be determined from the distribution of the eigenvalues and the matrix elements $[\tilde{\mathbf{A}}_0]_{0,p}$, $[\tilde{\mathbf{A}}_0^*]_{p,0}$. Its explicit form is not important. It is sufficient to assume its analyticity. Deforming the integration paths suitably, we integrate (3.22) by the method of steepest descent. It follows that

$$\begin{aligned}1/\xi=&\,2\sqrt{2}\cos\left(\theta+\frac{\pi}{4}\right)\ln\left|k^{1/2}\mathrm{snh}\left[\frac{2K}{\pi}(u-\Theta_{\mathrm{s}})\right]\right|\\&+2\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)\ln\left|k^{1/2}\mathrm{snh}\left[\frac{2K}{\pi}\left(-\frac{\lambda}{2}-\Theta_{\mathrm{s}}\right)\right]\right|\end{aligned} \tag{3.23}$$

where the saddle point Θ_{s} is determined as a function of θ by

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\Theta}\ln k^{1/2}\mathrm{snh}\left[\frac{2K}{\pi}(u-\Theta)\right]\\&\quad+\tan\left(\theta+\frac{\pi}{4}\right)\frac{\mathrm{d}}{\mathrm{d}\Theta}\ln k^{1/2}\mathrm{snh}\left[\frac{2K}{\pi}\left(-\frac{\lambda}{2}-\Theta\right)\right]=0\qquad\Theta=\Theta_{\mathrm{s}}\end{aligned} \tag{3.24}$$

with the condition

$$\Theta_{\mathrm{s}}=(\pi/2)\mathrm{i}+u\qquad\theta=3\pi/4 \tag{3.25}$$

(see figure 4).

When $u=0$, the interactions are isotropic. In this case (3.23)–(3.25) with $\theta=\pi/2$ reproduce the result by Buffenoir and Wallon [38]. At the first-order transition point, it is expected that the correlation length of the disordered phase is different from that of the ordered phase. Comparing their result with numerical calculations in [47, 48], Buffenoir and Wallon suggested that the correlation length ξ_{d} of the disordered phase is twice as large as the correlation length ξ_{o} of the ordered phase:

$$\xi_{\mathrm{d}}=2\xi_{\mathrm{o}} \tag{3.26}$$

for $u=0$ and $\theta=\pi/2$. The correlation length calculated in this subsection corresponds to ξ_{d} of the disordered phase (see also [49]).

3.2. Anisotropic interfacial tension and equilibrium crystal shape

We draw the ECS of the Potts model in the X–Y plane. At the first-order transition point two types of interfacial tension are possible: the interfacial tension γ_{oo} between two ordered phases and the interfacial tension γ_{do} between the disordered phase and an ordered phase [33, 49]. Here, we suppose that a droplet of an ordered phase whose volume (or area) is

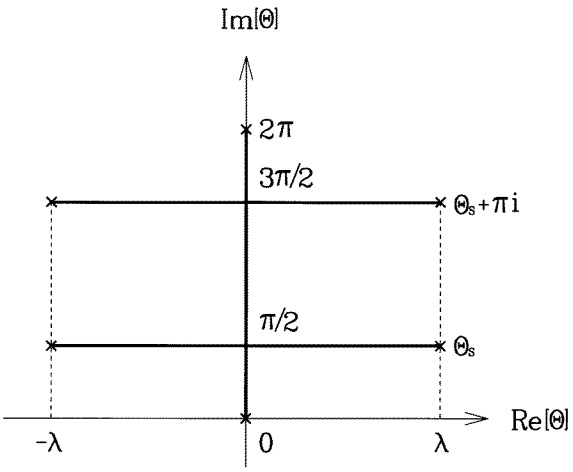

Figure 4. Due to a continuous distribution of the next-largest eigenvalues, the summation in (3.21) is converted into an integral over the imaginary parameters Θ_1 and Θ_2. When m becomes large with the ratio n/m fixed, the integral is estimated by the method of steepest descent. For each parameter there are two saddle points Θ_s and $\Theta_s + \pi i$. The contributions from the two saddle points differ only by a phase factor $(-1)^{m+n}$. As θ varies, Θ_s moves on the line $\mathrm{Im}(\Theta) = \pi/2$.

fixed is embedded inside a sea of another ordered phase. The ECS is defined as the shape of the minimum interface free energy. From the anisotropic interfacial tension γ_{oo} of the order–order interface, the ECS is derived by the use of the Wulff construction [1–4]

$$\Lambda X = \cos\theta\gamma_{oo} - \sin\theta(d\gamma_{oo}/d\theta) \tag{3.27a}$$

$$\Lambda Y = \sin\theta\gamma_{oo} + \cos\theta(d\gamma_{oo}/d\theta) \tag{3.27b}$$

where Λ is a scale factor adjusted to yield the volume of the crystal.

At the first-order transition point (2.18) becomes

$$\gamma_{oo}/k_B T = 1/\xi_d. \tag{3.28}$$

The anisotropic correlation length ξ_d calculated in 3.1 gives the anisotropic interfacial tension γ_{oo} via the relation (3.28). Using (3.23)–(3.25), and choosing the scale factor Λ suitably, we obtain

$$\Lambda(X - Y)/k_B T = \ln\left|k^{1/2}\mathrm{snh}\left[\frac{2K}{\pi}(u - \Theta_s)\right]\right| \tag{3.29a}$$

$$\Lambda(X + Y)/k_B T = \ln\left|k^{1/2}\mathrm{snh}\left[\frac{2K}{\pi}\left(-\frac{\lambda}{2} - \Theta_s\right)\right]\right|. \tag{3.29b}$$

As Θ_s moves from $-\lambda + (\pi/2)i$ to $\lambda + (\pi/2)i$ on the line $\mathrm{Im}(\Theta_s) = \pi/2$ (figure 4), (X, Y) sweeps out the ECS. The ECSs for $K_1/K_2 = 1$ and 1.6 are shown in figure 5.

We can rewrite (3.29) into the symmetric biquadratic relation (1.1) with α and β replaced by

$$\alpha = \exp[-\Lambda(X + Y)/k_B T] \qquad \beta = \exp[-\Lambda(X - Y)/k_B T]. \tag{3.30}$$

The coefficients A_3 and A_4 are given by

$$A_3 = \frac{1}{k\,\mathrm{snh}^2[2K(u + \lambda/2)/\pi]} \tag{3.31a}$$

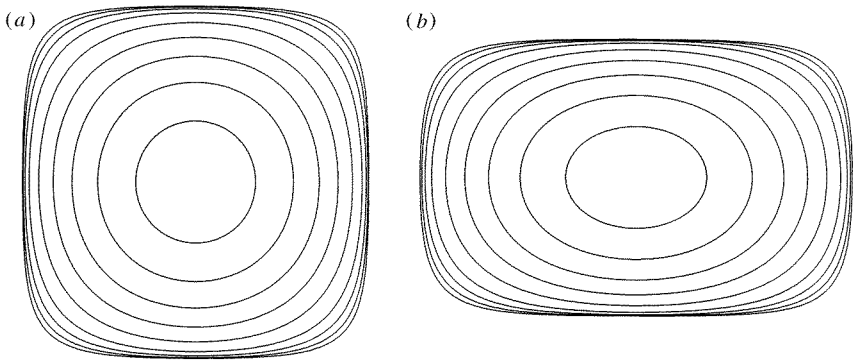

Figure 5. The ECSs of the Potts model at the first-order transition point. (*a*) The interactions are isotropic (or $K_1 = K_2$), and (*b*) $K_1/K_2 = 1.6$. From the outermost figure, $Q = 10^8$, 10^7, 10^6, 10^5, 10^4, 2000, 600, 200, 60, successively. Each figure is suitably scaled.

$$A_4 = -\frac{2\mathrm{cnh}[2K(u+\lambda/2)/\pi]\mathrm{dnh}[2K(u+\lambda/2)/\pi]}{k\,\mathrm{snh}^2[2K(u+\lambda/2)/\pi]} \tag{3.31b}$$

where the snh function is defined by (3.13) and

$$\mathrm{cnh}(v) = \mathrm{cn}(\mathrm{i}v) \qquad \mathrm{dnh}(v) = \mathrm{dn}(\mathrm{i}v). \tag{3.32}$$

Compare (3.29)–(3.31) with (4.20)–(4.23) of [26]. If the coordinate axes are rotated through $\pi/4$, the ECS derived here is identical to that of the eight-vertex model with

$$\lambda^{(8\mathrm{v})} = \lambda \qquad u_0^{(8\mathrm{v})}/2 = u \qquad I^{(8\mathrm{v})} = \pi/2 \tag{3.33}$$

where the variables $\lambda^{(8\mathrm{v})}$, $u_0^{(8\mathrm{v})}$, and $I^{(8\mathrm{v})}$ of the eight-vertex model are defined by (2.1), (2.2) of [26]. Note that, when $I^{(8\mathrm{v})} = \pi/2$, the eight-vertex model reduces to the six-vertex model. The results in sections 3.1 and 3.2 are consistent with the equivalence between the eigenvalue spectra of the transfer matrices of the Potts model and the six-vertex model shown in [38].

4. Associated spin chain and $E_q(1,1)$ algebra

Generally, for a two-dimensional classical system, we can define a one-dimensional quantum Hamiltonian which is related to the transfer matrix of the classical system [29, 43]. KSZ considered a quantum spin chain associated with the IRF model in section 3.1 (see [37, appendix B]). Analyses of the quantum spin chain give some useful insights for understanding physical meanings of the algebraic curve (1.1). In this section we investigate a connection between the algebraic curve (1.1) and the energy–momentum excitations of the spin chain.

KSZ defined the Hamiltonian $\mathbf{H}$ of the spin chain by

$$\mathbf{V}(u)/\mathbf{V}(-\lambda/2) \sim \mathbf{I} - (u+\lambda/2)\mathbf{H} + \cdots \qquad u \sim -\lambda/2 \tag{4.1}$$

where $\mathbf{I}$ is the identity operator. By the use of the limiting function $L(u)$ introduced by (3.11), the energy-momentum excitations are represented as

$$E - E_0 = -(\ln L)'(-\lambda/2) = \sum_j \varepsilon_j \tag{4.2a}$$

$$P - P_0 = -\mathrm{i}\ln L(-\lambda/2) = \sum_j p_j. \tag{4.2b}$$

It is noted that all the excitations are given by superpositions of free states; ε_j and p_j are the energy and the momentum of a free particle, respectively.

The ECS (1.1) is connected with the dispersion relation for free particles. Using (3.31) in (1.1*a*), we find that

$$(\alpha\beta + \alpha^{-1}\beta^{-1})k\,\mathrm{snh}^2[2K(u+\lambda/2)/\pi] + (\alpha\beta^{-1} + \alpha^{-1}\beta) \\ -2\mathrm{cnh}[2K(u+\lambda/2)/\pi]\mathrm{dnh}[2K(u+\lambda/2)/\pi] = 0. \tag{4.3}$$

To relate α and β to ε and p, substitute (3.29) into (3.30) and replace the saddle point Θ_s by an imaginary parameter Θ. It follows that

$$\alpha^{-1} = (-k)^{1/2}\mathrm{snh}\left[\frac{2K}{\pi}\left(-\frac{\lambda}{2} - \Theta\right)\right] \tag{4.4a}$$

$$\beta^{-1} = (-k)^{1/2}\mathrm{snh}\left[\frac{2K}{\pi}(u - \Theta)\right]. \tag{4.4b}$$

We expand β around the point $u = -\lambda/2$ to obtain

$$\alpha\beta^{-1} \sim 1 - (u + \lambda/2)\varepsilon + \cdots \qquad \alpha\beta \sim \mathrm{e}^{2\mathrm{i}p} + \cdots. \tag{4.5}$$

Similarly, the elliptic functions in (4.3) are expanded around the point $u = -\lambda/2$. From the coefficients of $(u+\lambda/2)^2$ in the expansion, we find the dispersion relation for free particles

$$\varepsilon^2 = (2K/\pi)^2[(1-k)^2 + 4k\sin^2 p]. \tag{4.6}$$

In the $k \to 1$ limit and for distances much larger than some microscopic length scale a (e.g. the lattice spacing), the spin chain should be equivalent to a relativistic field theory. To see this, we renormalize ε and p as

$$\bar{\varepsilon} = \pi\varepsilon/2aKk^{1/2} \qquad \bar{p} = 2p/a. \tag{4.7}$$

Substitute (4.7) into (4.6) and take the $k \to 1$ and $a \to 0$ limit with the ratio $(1-k)/a$ fixed. It follows that

$$\bar{\varepsilon}^2 = M^2 + \bar{p}^2 \tag{4.8}$$

where the mass term M is defined by

$$(1-k)/a = M \tag{4.9}$$

(see also [50]). The system is invariant under the transformations of the $(1+1)$ pseudo-Euclidian group. Because the pseudo-Euclidian group includes two translations and one boost, it is a three-parameter group. We denote the generators of the translations by $\mathbf{P}_0$ and $\mathbf{P}$, and the generator of the boost by $\mathbf{J}$. Relations among the generators are given by

$$[\mathbf{P}_0, \mathbf{P}] = 0 \qquad [\mathbf{J}, \mathbf{P}_0] = \mathrm{i}\mathbf{P} \qquad [\mathbf{J}, \mathbf{P}] = \mathrm{i}\mathbf{P}_0. \tag{4.10}$$

The mass squared $\mathbf{P}_0^2 - \mathbf{P}^2$ is the Casimir operator. Equation (4.8) is regarded as a realization of the Casimir invariant in the momentum representation.

Recently, Bonechi *et al* [51] proposed the q-deformed pseudo-Euclidian algebra $E_q(1,1)$ as a symmetry of discrete systems. They considered the kinematical symmetry of phonon propagation in harmonic crystals by interpreting the lattice spacing as a deformation parameter: $q = \mathrm{e}^{\mathrm{i}a}$. We note that $\mathbf{P}$ is determined up to an integer multiple of $2\pi/a$. Instead of $\mathbf{P}$, $\mathbf{K}(= \mathrm{e}^{\mathrm{i}a\mathbf{P}})$ appears as an element of algebra. The defining relations of $E_q(1,1)$ are

$$\mathbf{K}\mathbf{P}_0\mathbf{K}^{-1} = \mathbf{P}_0 \qquad \mathbf{K}\mathbf{J}\mathbf{K}^{-1} = \mathbf{J} + a\mathbf{P}_0 \\ \mathbf{K}\mathbf{K}^{-1} = \mathbf{I} \qquad [\mathbf{J}, \mathbf{P}_0] = (\mathbf{K} - \mathbf{K}^{-1})/2a. \tag{4.11}$$

The Casimir operator is deformed into

$$\mathbf{P}_0^2 - (2/a)^2 \sin^2(a\mathbf{P}/2). \tag{4.12}$$

The argument by Bonechi *et al* [51] is applicable to excitations of the spin chain defined by (4.1). In the momentum representation

$$\begin{aligned}\mathbf{J} &= i\bar{\varepsilon}\partial_{\bar{p}} + \frac{1}{2a}(\mathrm{e}^{\mathrm{i}a\bar{p}} - \mathrm{e}^{-\mathrm{i}a\bar{p}})\partial_{\bar{\varepsilon}} \\ &= \mathrm{i}\bar{\varepsilon}\partial_{\bar{p}} + \frac{1}{2a}(\bar{K} - \bar{K}^{-1})\partial_{\bar{\varepsilon}}\end{aligned} \tag{4.13}$$

where $\bar{\varepsilon}$ (respectively $\bar{p}$, $\bar{K}$) is the eigenvalue of $\mathbf{P}_0$ (respectively $\mathbf{P}$, $\mathbf{K}$). We can regard (4.6) with (4.7) as a realization of the deformed Casimir operator (4.12). The parametrization in terms of the Jacobian elliptic functions plays an essential role here. Using the elliptic parametrization, we obtain

$$\bar{\varepsilon} = -\frac{\mathrm{cnhdnh}}{ak^{1/2}\mathrm{snh}}\left[\frac{2K}{\pi}\left(-\frac{\lambda}{2} - \Theta\right)\right] \tag{4.14a}$$

$$\bar{K}^{-1} = \mathrm{e}^{-\mathrm{i}a\bar{p}} = -k\,\mathrm{snh}^2\left[\frac{2K}{\pi}\left(-\frac{\lambda}{2} - \Theta\right)\right] \tag{4.14b}$$

$$\mathbf{J} = \frac{\pi}{4Kk^{1/2}}\partial_\Theta. \tag{4.14c}$$

Eliminating Θ in (4.14*a*) and (4.14*b*) derives the Casimir invariant (4.12). Thus, in the special limit, the algebraic curve (1.1) is closely connected with the symmetry of the spin chain.

5. Summary and discussion

In this paper the exact ECS of the Potts model was calculated for $Q > 4$ at the first-order transition point. In section 2 we considered duality transformations which connect the high-temperature disordered phase with low-temperature ordered phases. For general Q we proved a simple thermodynamic relation between the anisotropic correlation length ξ_{d} of the disordered phase and the anisotropic interfacial tension γ_{oo} of the order–order interface. In section 3 analyses were restricted to the case $Q > 4$, where the Potts model has a first-order transition point. We calculated the anisotropic correlation length ξ_{d} at the first-order transition point. Then, from ξ_{d}, the ECS was found with the help of the thermodynamic relation and the Wulff construction. The ECS was represented in terms of the algebraic curve (1.1). Regarding Q as a temperature scale, we showed that the Potts model has the same ECS as the eight-vertex model.

In section 4 we discussed a relation between the algebraic curve (1.1) and the energy–momentum excitations of a spin chain which is associated with the Potts model. We found that (1.1) reflects the symmetry of the spin chain (or the $E_q(1,1)$ algebra) in a special limit. There, all the excitations are represented as superpositions of free particles. In the analyses of the XYZ spin chain bound states appear besides free states; see [50, 52, 53]. We can show that the energy–momentum excitations of a bound state is connected with the coproducts of the $E_q(1,1)$ algebra. From (4.1) and the parametrization in terms of the Jacobian elliptic functions, it is suggested that (1.1) is related to a symmetry of systems which are discretized in both directions. Detailed investigations about these problems will be reported separately.

In the calculation of ξ_d, we estimated an integral over the band of next-largest eigenvalues by the method of steepest descent. Note that in this calculation the periodicity of $L(u)$ played important roles: the periodicity of the snh function in $L(u)$ along the imaginary axis allowed us to deform the integration paths suitably; we found that, as θ varies from 0 to 2π, the saddle point Θ_s moves on the line $\mathrm{Im}(\Theta_s) = \pi/2$ by 2λ; thus, the periodicity along the real axis is needed to determine ξ_d uniquely for all directions (figure 4). We discuss the anisotropic correlation length ξ_d at $T = T_f + \epsilon$; T_f is the first-order transition temperature and $\epsilon(> 0)$ is an infinitesimal parameter. We cannot define commuting transfer matrices there. The limiting functions for the transfer matrix and the shift operator are denoted by L_t and L_s, respectively. The continuity of ξ_d around $T = T_f$ suggests that ξ_d is still determined by integrating L_t and L_s in a similar way to the calculation at $T = T_f$. This fact implies that L_t and L_s are represented by the use of doubly periodic functions. Combining the relation to the dispersion curve (4.6) for free particles, we expect that, for $T > T_f$, L_t and L_s will also be written in the form of (3.12), which means that ξ_d will be represented by the use of an algebraic curve similar to (1.1). To clarify this point, numerical calculations are now in progress.

Acknowledgments

I would like to express my appreciation to Professor M Kikuchi for useful discussion about the Potts model. This work was partially supported by the Grant-in-Aid for Encouragement of Young Scientists from the Ministry of Education, Science and Culture (no 07854019).

References

[1] Wulff G 1901 *Z. Krist. Mineral.* **34** 449–530
[2] von Laue M 1944 *Z. Krist. Mineral.* **105** 124–33
[3] Herring C 1951 *Phys. Rev.* **82** 87–93
[4] Burton W K, Cabrela N and Frank F C 1951 *Phil. R. Soc.* A **243** 229–358
[5] van Beijeren H 1977 *Phys. Rev. Lett.* **38** 993–6
[6] Jayaprakash C, Saam W F and Teitel S 1983 *Phys. Rev. Lett.* **50** 2017–20
[7] Rottman C and Wortis M 1981 *Phys. Rev.* B **24** 6274–7
[8] Avron J E, van Beijeren H and Zia R K P 1982 *J. Phys. A: Math. Gen.* **15** L81–6
[9] Zia R K P and Avron J E 1982 *Phys. Rev.* B **25** 2042–5
[10] Zia R K P 1986 *J. Stat. Phys.* **45** 801-13
[11] Holtzer M 1990 *Phys. Rev. Lett.* **64** 653–6
[12] Holtzer M 1990 *Phys. Rev.* B **42** 10 570-82
[13] Akutsu Y and Akutsu N 1990 *Phys. Rev. Lett.* **64** 1189–92
[14] Müller-Krumbhaar H 1979 *Monte Carlo Methods in Statistical Physics* ed K Binder (Berlin: Springer)
Saito Y and Müller-Krumbhaar H 1984 *Applications of the Monte Carlo Method in Statistical Physics* ed K Binder (Berlin: Springer)
[15] Carmi Y, Lipson S G and Polturak E 1987 *Phys. Rev.* B **36** 1894–8
[16] Fink R, Krausch G, Luckscheiter B, Platzer R, Wöhrmann U and Schatz G 1993 *Phys. Rev. Lett.* **70** 2455–8
[17] Watson P G 1972 *Phase Transitions and Critical Phenomena* vol 2, ed C Domb and M S Green (London: Academic)
[18] Fradkin E, Huberman B A and Shenker S H 1978 *Phys. Rev.* B **18** 4789–814
[19] Zia R K P 1978 *Phys. Lett.* **64A** 345–7
[20] Kramers H A and Wannier G K 1941 *Phys. Rev.* **60** 252–62
[21] Wegner F G 1971 *J. Math. Phys.* **12** 2259–72
[22] Greenberg W 1973 *Commun. Math. Phys.* **29** 163–75
Greenberg W 1977 *J. Math. Phys.* **18** 1985–6
[23] Wu F Y and Wang Y K 1976 *J. Math. Phys.* **17** 439–40
[24] Cheng H and Wu T T 1967 *Phys. Rev.* **164** 719–35

[25] McCoy B M and Wu T T 1973 *The Two-Dimensional Ising Model* (Cambridge, MA: Harvard University Press)
[26] Fujimoto M *J. Stat. Phys.* 1992 **67** 123–54
[27] Fujimoto M *J. Stat. Phys.* 1990 **59** 1355–81
[28] Fujimoto M *J. Stat. Phys.* 1990 **61** 1295–304
[29] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[30] Baxter R J and Pearce P A 1983 *J. Phys. A: Math. Gen.* **16** 2239–55
[31] Pearce P A 1985 *J. Phys. A: Math. Gen.* **18** 3217–26
[32] Suzuki J 1989 *J. Phys. Soc. Japan* **58** 3111–22
[33] Fujimoto M 1993 *J. Phys. A: Math. Gen.* **26** 2285–99
[34] Potts R B 1952 *Proc. Camb. Phil. Soc.* **48** 106–9
[35] Wu F Y 1982 *Rev. Mod. Phys.* **54** 235–68
[36] Baxter R J 1973 *J. Phys. C: Solid State Phys.* **6** L445–8
[37] Klümper A, Schadschneider A and Zittartz J 1989 *Z. Phys.* B **76** 247–58
[38] Buffenoir E and Wallon S 1993 *J. Phys. A: Math. Gen.* **26** 3045-62
[39] Laanait L 1987 *Phys. Lett.* **124A** 480–4
[40] Bricmont J, Lebowitz J L and Pfister C E 1980 *Ann. N. Y. Acad. Sci* **337** 214–223
[41] Ginibre J 1970 *Commun. Math. Phys.* **16** 310–28
[42] Griffith R B 1971 *Les Houches Lectures, 1970* ed C De Witt and R Stora (New York: Gordon and Breach)
[43] Wadati M and Akutsu Y 1988 *Prog. Theor. Phys. Suppl.* **94** 1–41
[44] Wadati M, Deguchi T and Akutsu Y 1989 *Phys. Rep.* **180** 247–332
[45] Baxter R J 1982 *J. Stat. Phys.* **28** 1–41
[46] Pearce P A 1987 *J. Phys. A: Math. Gen.* **20** 6463–9
[47] Peczak P and Landau D P 1989 *Phys. Rev.* B **39** 11 932–42
[48] Gupta S and Irbäck A 1992 *Phys. Lett.* B **286** 112–7
[49] Borgs C and Janke W 1992 *J. Physique* **2** 2011–8
[50] Klümper A and Zittartz J 1988 *Z. Phys.* B **71** 495–507
[51] Bonechi F, Celeghini E, Giachetti R, Sorace E and Tarlini M 1992 *Phys. Rev. Lett.* **68** 3718–20
[52] Johnson J D, Krinsky S and McCoy B M 1973 *Phys. Rev.* A **8** 2526–47
[53] Fujimoto M 1996 *Physica* **233A** 485–502